Linearization - Advantages and Shortcomings Toward Control System Design

Muhammad Nizam Kamarudin[1*], Sahazati Md. Rozali[2], Mohd Hendra Hairi[3], Alias Khamis[4], Abdul Rashid Husain[5]
[1,2,3,4] Center for Robotics and Industrial Automation (CeRia)
[1,3,4] Faculty of Electrical Engineering, Universiti Teknikal Malaysia Melaka, Hang Tuah Jaya, 76100 Durian Tunggal, Melaka, Malaysia.
[2] Faculty of Electrical Engineering Technology, Universiti Teknikal Malaysia Melaka, Hang Tuah Jaya, 76100 Durian Tunggal, Melaka, Malaysia.
[5] Faculty of Electrical Engineering, Universiti Teknologi Malaysia, 81310 Skudai, Johor, Malaysia.
[*]corresponding authors: nizamkamarudin@utem.edu.my

Abstract – *This brief manuscript discusses the necessity to linearize nonlinear systems. Thorough review on nonlinear phenomena in dynamical and numerical system is presented. The methodology to linearize nonlinear system in Jacobian approach is shown in didactic manner. Numerical and dynamical example of nonlinear system is provided to enhance understanding. Afterward, the comparison between both linearized and non-linearized system is literally discussed. The outcomes concluded that linearization process is a linear approximation of a nonlinear system that is only valid in a small region around an operating point.*

Keywords: *Jacobian, Linear system, nonlinear system, equilibrium point, Linearization*

Article History
Received 29 January 2019
Received in revised form 9 May 2019
Accepted 9 May 2019

I. Introduction

Many physical systems are nonlinear in nature. Engineers involve in multifarious technologies ranging from renewable energy [1]–[2], electro-mechanical systems [3], manufacturing industries [4]–[5], robotics [6]–[7], aerospace [8], maritime [9] and automotive sectors [10] are dealing with nonlinear phenomena. In μ-nano systems, nonlinearity comes from hysteresis phenomena, friction, as well as discontinuous behavior [11]. Robotic systems hold nonlinearity due to Sine or Cosine functions. Whereas, study in wind turbine technology shows that the power coefficient characteristic of typical wind turbine system behaves as a nonlinear function [12]–[13]. Likewise, backlash appears in rotational geared mechanical system introduces nonlinearity phenomena [14]. If the backlash is accounted in the system dynamics, the system model behaves as nonlinear function which is difficult to control. Practically, the backlash is known to be insignificant to the system dynamics and therefore being neglected by the control designer.

Nonlinear systems do not fulfilll superposition principle as linear systems do. Nonlinear systems absorb nonlinear phenomena such as chaos, limit cycle, saturation, finite escape time. Nonlinear systems have multiple isolated equilibrium points, or sometimes infinite and even not exist. If the nonlinearity is too large and significant to the system dynamics, the performance of the system may be deteriorated when the system is controlled by linear controller derived through linearized model [15]. The controllability and observability of nonlinear system is also hard to prove. The stability can be proved by using high level mathematical manipulation such as Lyapunov [16]–[18] or Popov [19]–[20] through Nyquist. However, frequency analysis for nonlinear system is almost impossible in order to facilitate Nyquist criteria. Thus, solving nonlinear systems requires advance control techniques. The presence of exogenous disturbances and uncertainties in the nonlinear systems dynamic is sometimes inevitable, and give catastrophic effect to the stability and robustness of closed loop systems.

Nonlinear phenomena sometimes necessary as adopted by oscillator and cyclic systems. Some strange behaviors of nonlinear systems can be observed in Van der Pol oscillator [21]–[22]. The oscillator is modeled as two-dimensional second order system

$$\dot{x}_1 = \dot{x}_2 \tag{1}$$

$$\dot{x}_2 = -x_1 \varepsilon h(x_1) x_2 \tag{2}$$

where

$$h(x_1) = -1 + x_1^2 \qquad (3)$$

gives nonlinearity to the system. Limit cycle effect in Van der Pol oscillator can be observed by tuning the non-negative parameter ε, as shown by the phase portrait in Fig. 1. Note that both x_1 and x_2 are system states.

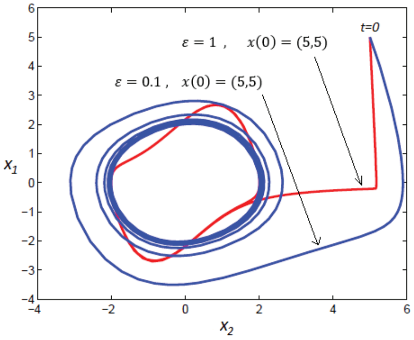

Fig. 1. Historical trajectory of x_1 and x_2

Previous literatures demonstrate methods to deal with nonlinear systems. For example, author in [18] deals with numerical second order nonlinear system with uncertainties and exogenous disturbances. The numerical nonlinear system of the form

$$\dot{x}_1 = x_2 + x_1^2 \sin 0.01t \qquad (4)$$

$$\dot{x}_2 = x_1^3 e^{x_2} u + e^{x_2} \cos(x_1 x_2) \qquad (5)$$

is stabilized by using advanced backstepping with nonlinear damping function. Another strange behavior of nonlinear effect can be observed in the tunnel diode [23], where the study discusses about bifurcation phenomena in tunnel diode circuit.

In linear systems, input/output frequency domain methods are known to be effective. However, in nonlinear systems, poles and zeros, frequency domain, phase and gain margin are not defined. Thus, solving nonlinear systems requires advanced control techniques. Linear systems can be stabilized by linear controllers such as pole placement approach [24]–[26], linear quadratic regulator (LQR) [27]–[29], linear quadratic Gaussian (LQG) [30], model reference adaptive control (MRAC) [7], proportional-integral-derivative (PID) technique [31]–[33] and many more. MRAC requires reference LTIV model. The error dynamic between actual model and reference model is processed by the adaptation law that can be designed by using gradient method, hyperstability or Lyapunov. Design steps for optimal control using LQR can be possible if the system matrix is linear because separation principle is valid.

II. Linearization

In some cases, linearization of nonlinear systems is normally obtainable by using the Jacobian matrix at equilibrium points [34]–[36]. Then, by using the linearized system, a simple linear controller can be applied to achieve stabilization. In some systems, linearization gives freedom to designer. For instance, designing feedback gain K for state feedback system $u(t) = -Kx(t)$ can be done by various techniques such as linear matrix inequality (LMI) and artificial computing.

Linearization via Jacobian can be the easiest technique available to linearize nonlinear function. For instance, let an autonomous system of the form

$$\frac{dx}{dt} = f(x,y) \qquad (6)$$

$$\frac{dy}{dt} = g(x,y) \qquad (7)$$

where $f(\cdot)$ and $g(\cdot)$ are continuously n-times differentiable C^n. Assume (x_0, y_0) be the equilibrium point for the system in equation (6) and equation (7), *Theorem 1* confirms the exponentially stable origin for (x_0, y_0).

Theorem 1

Let $x = 0$ be an equilibrium point for the nonlinear system

$$\frac{dx}{dt} \triangleq \dot{x} = f(t,x) \qquad (8)$$

where t represents time and $f: [0,\infty) \times \mathcal{D} \longrightarrow \mathcal{R}^n$ is continuously differentiable, $\mathcal{D} = \{x \in \mathcal{R}^n | \|x\|_2 < r\}$, and the Jacobian matrix $\mathcal{J} \triangleq \left[\frac{\partial f}{\partial x}\right]$ is bounded and Lipschitz on $\mathcal{D}$, uniformly in t. Let

$$A(t) = \frac{\partial f}{\partial x}(t,x)\Big|_{x=0} \qquad (9)$$

then the origin is an exponentially stable equilibrium point for the nonlinear system if it is an exponentially stable equilibrium point for the linear system

$$\dot{x} = A(t)x \qquad (10)$$

$\cdots$ *End of Theorem* 1 $\cdots$

With *Theorem 1*, the Jacobian at (x_0, y_0) can be computed as

$$\begin{aligned} \frac{dx}{dt} &= f(x,y) \\ &\approx f(x_0,y_0) + \frac{\partial f}{\partial x}(x_0,y_0)(x - x_o) \\ &\quad + \frac{\partial f}{\partial x}(x_0,y_0)(y - y_o) \end{aligned} \qquad (11)$$

and

$$\begin{aligned}\frac{dy}{dt} &= g(x,y)\\ &\approx g(x_0,y_0)+\frac{\partial g}{\partial x}(x_0,y_0)(x-x_o)\\ &\quad+\frac{\partial g}{\partial x}(x_0,y_0)(y-y_o)\end{aligned} \tag{12}$$

since (x_0, y_0) is the equilibrium point, $f(x_0, y_0) \Rightarrow 0$ and, $g(x_0, y_0) \Rightarrow 0$. As such, the equilibrium points (x_0, y_0) implies

$$f(x_0,y_0) = g(x_0,y_0) = 0 \tag{13}$$

Therefore, the linearized system (6)-(7) can be concluded as in equation (14) and equation (15).

$$\frac{dx}{dt} = \frac{\partial f}{\partial x}(x_0,y_0)(x-x_o) + \frac{\partial f}{\partial y}(x_0,y_0)(y-y_o) \tag{14}$$

$$\frac{dy}{dt} = \frac{\partial g}{\partial x}(x_0,y_0)(x-x_o) + \frac{\partial g}{\partial y}(x_0,y_0)(y-y_o) \tag{15}$$

Thus, Jacobian matrix $\mathcal{J}$ of system (6)-(7) at (x_0, y_0) can be written as

$$\mathcal{J} = \begin{bmatrix} \frac{\partial f}{\partial x}(x_0,y_0)(x-x_o) & \frac{\partial f}{\partial y}(x_0,y_0)(y-y_o) \\ \frac{\partial g}{\partial x}(x_0,y_0)(x-x_o) & \frac{\partial g}{\partial y}(x_0,y_0)(y-y_o) \end{bmatrix} \tag{16}$$

III. Numerical Illustration

This section illustrates two cases nonlinear systems. First, dynamical physical system will be linearized in Jacobian. Secondly, the Jacobian will be exploited to linearize the numerical nonlinear function.

A. Nonlinear Pendulum

To demonstrate the approach in equation (11) - (16), consider the dynamics of pendulum system in

$$\ddot{\theta} = -a\sin\theta - b\dot{\theta} + cT \tag{17}$$

The input to the system is the torque applied to the pendulum. In steady state, the dynamics in equation (17) appears as

$$-a\sin\theta + cT_{ss} = 0 \tag{18}$$

Then, stabilizing the pendulum at $\theta = \delta$ defines two state variables $x_1 = \theta - \delta$ and $x_2 = \dot{\theta}_n$. As such, $\dot{x}_1 = x_2$ and , $\dot{x}_2 = -a[\sin(x_1+\delta) - \sin\delta] - bx_2 + cu$, where u represents the input (torque) to the pendulum. Hence, linearize the system at origin renders $\left.\frac{\partial x_1}{\partial x_1}\right|_{(0,0)} = 0$, $\left.\frac{\partial x_1}{\partial x_2}\right|_{(0,0)} = 1$, $\left.\frac{\partial x_2}{\partial x_1}\right|_{(0,0)} = -a\cos\delta$, and $\left.\frac{\partial x_2}{\partial x_2}\right|_{(0,0)} = -b$. Therefore, the linearized pendulum system at origin (0,0) can be written as

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix} = \begin{bmatrix}0 & 1\\ -a\cos\delta & -b\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix} + \begin{bmatrix}0\\ c\end{bmatrix}u \tag{19}$$

where $A = \begin{bmatrix}0 & 1\\ -a\cos\delta & -b\end{bmatrix}$ is the system matrix and $B = [0 \quad c]^T$ is the input matrix. The pair (A, B) is controllable. Taking $K = [K_1 \quad K_2]$ such that $A - BK$ is Hurwitz will guarantee the stability of the pendulum system. Parameter K can be formulated via state-feedback control techniques that is not within the scope of this studies.

B. Numerical Nonlinear System

Consider nonlinear system

$$\dot{x} = y \tag{20}$$

$$\dot{y} = (1-x^2)y + x \tag{21}$$

The system is autonomous as no stimulus is exerted to the system. The nonlinearity comes from x^2y-term that appear in the 2nd-subsystem. The equilibrium point of equation (20) and equation (21) is $(x,y) = (0,0)$. Therefore, the Jacobian matrix is formulated as

$$\mathcal{J} = \begin{bmatrix} \frac{\partial}{\partial x}(y) & \frac{\partial}{\partial y}(y) \\ \frac{\partial}{\partial x}((1-x^2)y+x) & \frac{\partial}{\partial y}((1-x^2)y+x) \end{bmatrix} \tag{22}$$

and yields

$$\mathcal{J} = \begin{bmatrix}0 & 1\\ -2xy+1 & 1-x^2\end{bmatrix} \tag{23}$$

Then, the Jacobian matrix at equilibrium can be computed as

$$\mathcal{J} = \begin{bmatrix}0 & 1\\ 1 & 1\end{bmatrix} \tag{24}$$

in order to get the linearized system at $(0,0)$ as

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix} = \begin{bmatrix}0 & 1\\ 1 & 1\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix} \tag{25}$$

In equation (25), both x_1 and x_2 are defined as system states. As a result, and for simplicity, one would obtain

$$\dot{x} = y \tag{26}$$

$$\dot{y} = y + x \tag{27}$$

If both linearized system (equation (26) – equation (27)) and actual system (equation (20) – equation (21)) are injected by step $u(t)$, the trajectory can be recorded as in Fig. 2.

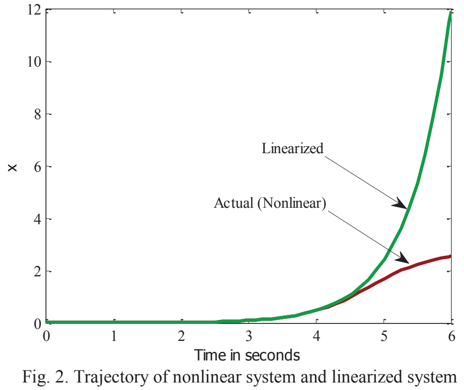

Fig. 2. Trajectory of nonlinear system and linearized system

IV. Discussion

Linearization relaxes the need of using nonlinear control techniques to stabilize nonlinear systems. However, control law that is designed by using linearized model would not be robust within the wide range of operation. This phenomenon can be realized in Fig. 2 where the trajectory of the linearized system departs away from the trajectory of the actual system when time approaching ∞. After 4.5 seconds, the result shown in Fig. 2 bears no trajectory resemblance between actual nonlinear system in equations (20)-(21) and its linearized version in equations (26)-(27). The contradiction between the actual nonlinear system and its linearized version in Fig. 2 would devise a cunning test to control engineers. As such, the controller that is designed based on linearized model will not be able to guarantee the stabilization beyond wide range of nonlinear sector when applied to nonlinear system.

V. Conclusion

Linearize or not to linearize a nonlinear function (or system) is highly depending on the objectives of the solution to a nonlinear system under studies. Researchers studying the chaotic or bifurcation phenomena possibly will not to linearize the system at hand because relaxing nonlinear phenomena will diminish the sources of bifurcation and chaotic in the system. In some control engineering fields, linearization is a must if the nonlinearity become insignificant and does not offer catastrophic effect to the system under studies. In this case, linearization is utilize only to ease the steps for controller design.

Acknowledgements

We acknowledge the Center for Robotics and Industrial Automation, Faculty of Electrical Engineering Universiti Teknikal Malaysia Melaka - UTeM for research facilities and research collaboration. Special appreciation also to the Faculty of Electrical Engineering, Universiti Teknologi Malaysia - UTM for research collaboration.

References

[1] F. Chiou, J. P. Gentle and T. R. McJunkin, Dispatchable renewable energy model for microgrid power system. *IEEE Conference on Technologies for Sustainability, pp.195-199, 2016.*

[2] Z. Cao, F. O'Rourke and W. Lyons, Performance modelling of a small-scale wind and solar energy hybrid system, *28th Irish Signals and Systems Conference, pp.1-6, 2017.*

[3] M. I. Ahmed, M. Y. Hazlina and M. M. Rashid, Mathematical Modeling and Control of Active Suspension System for a Quarter Car Railway Vehicle, *Malaysian Journal of Mathematical Sciences, Vol. 10(S), pp.227-241, 2016.*

[4] Jiwang Du, Qichang He and Xiumin Fan, Automating generation of the assembly line models in aircraft manufacturing simulation, *IEEE International Symposium on Assembly and Manufacturing, pp.155-159, 2013.*

[5] Wen-Chyuan Chiang, Timothy L. Urban and Chunyong Luo, Balancing stochastic two-sided assembly lines, *International Journal of Production Research, Vol. 54 (Issue 20), pp.6232-6250, 2016.*

[6] H. M. Al-Qahtani, Amin A. Mohammed and M. Sunar, Dynamics and Control of a Robotic Arm Having Four Links, *Arabian Journal for Science and Engineering, Vol. 42 (Issue 5), pp.1841-1852, 2017.*

[7] A. Kirecci, M. Topalbekiroglu and I. Eker, Experimental evaluation of a model reference adaptive control for a hydraulic robot: A case study, *Robotica, Vol.21 (Issue 1): pp.71-78, 2003.*

[8] S. Roggia, F. Cupertino, C. Gerada and M. Galea, Axial Position Estimation of Conical Shaped Motors for Aerospace Traction Applications. *IEEE Transactions on Industry Applications, Vol. 53 (Issue 6): pp.5405-5414, 2017.*

[9] A. Witkowska, M. Tomera and R. mierzchalski, A Backstepping Approach to Ship Course Control, *International Journal of Applied Mathematics and Computer Science, Vol. 17 (Issue 1): pp.73-85, 2007.*

[10] Yanjun Huang, Amir Khajepour, Farshid Bagheri and Majid Bahrami, Modelling and optimal energy-saving control of automotive air-conditioning and refrigeration systems, *Journal of Automobile Engineering, Vol. 231 (Issue 3): pp.291-309, 2016.*

[11] Kis Andras, Physical models for micro and nano systems, Cole polytechnique fdrale de Lausanne, Course books 2017-2018.

[12] Muhammad Nizam Kamarudin, Abdul Rashid Husain and Mohamed Noh Ahmad, Robust bounded control for uncertain nonlinear systems: application to a nonlinear strict feedback wind turbine model with explicit wind speed dynamics, *Journal of Theoretical and Applied Information Technology, Vol. 63 (Issue 3): pp.718-732, 2014.*

[13] Nor Syaza Farhana Mohamad Murad, Muhammad Nizam Kamarudin, Sahazati Md Rozali and Mohd Hendra Hairi, Achieving optimum tip-speed-ratio of a two-mass wind turbine system, *IEEE International Conference on Power and Energy (PECon), pp.757-762, 2016.*

[14] Nouby M. Ghazaly, Ali A. Kamel and M. O. Mousa, Influence of Misalignment and Backlash on Spur Gear using FEM, *International Journal of Mechanical and Production Engineering, Vol. 2 (Issue 12), pp.2320-2092, 2014.*

[15] Muhammad Nizam Kamarudin, Abdul Rashid Husain and Mohamed Noh Ahmad, Variable Speed Wind Turbine with External Stiffness and Rotor Deviation Observer, *Applied Mechanics and Materials, Vol. 661, pp.154-159, 2014.*

[16] Zheng Wang, Xiaoping Liu, Kefu Liu, Shuai Li and Huanqing Wang, Backstepping-Based Lyapunov Function Construction using Approximate Dynamic Programming and Sum of Square

Techniques, *IEEE Transactions On Cybernetics, Vol. 47 (Issue 10), pp.3393- 3403, 2017*.

[17] Tushar Kanti Roy, Md. Apel Mahmud, Weixiang Shen, and Amanullah Maung Than Oo, Nonlinear Adaptive Excitation Controller Design for Multimachine Power Systems with Unknown Stability Sensitive Parameters, *IEEE Transactions On Control Systems Technology, Vol. 25 (Issue 6), pp.2060-2072, 2017*.

[18] Muhammad Nizam Kamarudin, Abdul Rashid Husain and Mohamed Noh Ahmad, Control of Uncertain Nonlinear Systems using Mixed Nonlinear Damping Function and Backstepping Techniques, *IEEE International Conference on Control System, Computing and Engineering*, *Penang, pp.105-109, 2012*.

[19] M. R. James, I. R. Petersen and V. Ugrinovskii, A Popov stability condition for uncertain linear quantum systems, *American Control Conference, pp. 2551-2555, 2013*.

[20] E. Jackson and J. Aggarwal, Popov and Hurwitz stability criterions: A comparison, *IEEE Transactions on Automatic Control, Vol. 11 (Issue 3), pp.623-24, 1996*.

[21] Xinye Lia, Huabiao Zhang and Lijuan Zhang, Response of the Duffng-van der Pol oscillator under position feedback control with two time delays, *Shock and Vibration, Vol. 18, pp. 377386, 2011*.

[22] Marios Tsatsos, *Theoretical and Numerical Study of the Van der Pol equation*, Ph.D Thesis Dissertation, Department of Physics,School of Sciences, Aristotle University of Thessaloniki, July 2016.

[23] Nikhil M. Kriplani, Stephen Bowyer, Jennifer Huckaby and Michael B. Steer, Modelling of an Esaki Tunnel Diode in a Circuit Simulator, *Active and Passive Electronic Components, Vol. 2011, pp.1-8, 2011*.

[24] Volker Mehrmann and Hongguo Xu, An analysis of the pole placement problem: The single-input case, *Electronic Transactions on Numerical Analysis, Vol.4, pp.89-105, 1996*.

[25] Yuan Gong Sun, LongWang, Guangming Xie and Mei Yu, Improved overshoot estimation in pole placements and its application in observer-based stabilization for switched systems, *IEEE Transactions on Automatic Control, vol. 51 (Issue 12), pp.1962-1966, 2006.*

[26] Kaiyang Yang and Robert Orsi, Generalized pole placement via static output feedback: A methodology based on projections, *Automatica, Vol. 42 (Issue1), pp.2143-2150, 2006*.

[27] Muhammad Nizam Kamarudin, Abdul Rashid Husain and Sahazati Md.Rozali, Observer-based output feedback control with linear quadratic performance, *Procedia Engineering, Vol.53 (), pp.233-240, 2013*.

[28] Jos_e B Mare and Jos_e A De Don, Solution of the input-constrained LQR problem using dynamic programming, *Systems & Control Letters, Vol. 56 (Issue 5), pp.342-348, 2007*.

[29] Alberto Bemporad, Manfred Morari, Vivek Dua and Efstratios N Pistikopoulos, The explicit linear quadratic regulator for constrained systems, *Automatica, Vol. 38 (Issue 1), :3-20, 2002*.

[30] Y. Eun, C. Gokcek, P. T. Kabamba and S. M. Merkov, An LQG approach to ystems with saturating actuators and anti-windup implementation, *Mathematical Problems in Engineering, Vol.8 (Issue 4-5):311-321, 2002*.

[31] M. N. Kamarudin, S. M. Rozali, M. H. Hairi, F. Hanaffi, M. S. M. Aras and M. K. M. Zambri, Realization of Real-Time Hardware-in-the-Loop for a Liquid Level with Open-loop Ziegler Nichols Technique, *International Journal of Electrical Engineering and Applied Sciences (IJEEAS), Vol.1(2), pp.47-52, 2018.*

[32] Kiam Heong Ang, Gregory Chong and Yun Li, PID control system analysis, design, and technology, *IEEE Transactions on Control Systems Technology, Vol.13 (Issue 4), pp.559-576, 2005*.

[33] Guillermo J. Silva, Aniruddha Datta and S. P. Bhattacharyya, New results on the synthesis of PID controllers, *IEEE Transactions on Automatic Control, Vol. 47 (Issue 2), pp.241-252, 2002.*

[34] Zhendong Sun and Xiaohua Xia, The explicit linear quadratic regulator for constrained systems. *Automatica, Vol. 33 (Issue 7) pp.1339-1344, 1997*.

[35] C. Daizhan, H. Xiaoming and W. Yuzhen, Non-regular feedback linearization of nonlinear systems via a normal form algorithm, *Automatica, Vol. 40, pp.439-447, 2004*.

[36] M. Ababneh, M. Salah and K. Alwidyan, Non-regular feedback linearization of nonlinear systems via a normal form algorithm, *Jordan Journal of Mechanical and Industrial Engineering, Vol. 5 (Issue 6), pp.567-571, 2011*.